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Maximal Subgroups of $GL_{2n}(K)$, $SL_{2n}(K)$, $PGL_{2n}(K)$ and $PSL_{2n}(K)$ Associated with Symplectic Polarities

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1. INTRODUCTION

Although considerable progress has been made in the representation theory of the classical groups, their subgroup structure is far from being determined. The only classical groups for which complete lists of maximal subgroups are known are certain finite classical groups of dimensions 4 or less: see [1; 3, p. 260; 7; 9–12].

The geometry of a classical group may be used not only to obtain subgroups but also to predict when a subgroup is maximal. Suppose that G is the group fixing a certain geometric configuration $\mathscr{C}$ situated in either a vector or a projective space. With each subconfiguration $\mathscr{F}$ of $\mathscr{C}$ is associated its stabilizer H in G; usually H is a proper subgroup of G. A simple example may illuminate: take $\mathscr{C}$ to be a sphere centred on the origin in ordinary real 3-dimensional Euclidean space, so that G is the orthogonal group $O_3(\mathbb{R})$, and take $\mathscr{F}$ to be a tangent line to the sphere. In this case $\mathscr{F}$ has an invariant subconfiguration $\mathscr{F}^*$, the point of contact of the tangent line, and H is a proper subgroup of the stabilizer in $G = O_3(\mathbb{R})$ of the point $\mathscr{F}^*$ and so is not a maximal subgroup of G. Similarly, in the general case one can only hope that H is maximal in G when $\mathscr{F}$ has no proper, nontrivial invariant subconfiguration; were such a configuration of $\mathscr{F}$ to exist then H would lie in its stabilizer. Even then one must not be too confident, for almost never does the geometry of $\mathscr{C}$ reveal all the subgroups of G. However, if $\mathscr{C}$ has a well-explored and rich geometry then one's expectation may be high!

In this paper we take $\mathscr{C}$ to be the set of all subspaces of a vector space and $\mathscr{F}$ to be the configuration of the totally isotropic 2-dimensional subspaces of a symplectic polarity. So let $B(x, y)$ be a nonsingular alternating bilinear form on a vector space V of dimension $2n \geqslant 4$ over some field K. Then $\mathscr{F}$ consists of those 2-dimensional subspaces on which the restriction of $B(x, y)$ is the zero form. A linear isomorphism A of V belongs to the group of symplectic similitudes

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$GSp_{2n}(K)$ if and only if there is a non-zero element λ in K such that [5, pp. 18, 19]

$$B(Ax, Ay) = \lambda B(x, y). \tag{1}$$

The A in $GSp_{2n}(K)$ for which $\lambda = 1$ form the usual symplectic group $Sp_{2n}(K)$. Since $\mathscr{F}$ is determined by any $\lambda B(x, y)$ for which $\lambda \neq 0$ the group of $\mathscr{F}$ is $GSp_{2n}(K)$: see Lemma 1 below. If one passes to the projective space $\mathscr{P}$ whose points are the 1-dimensional subspaces of V then the image of $\mathscr{F}$ consists of the lines of a nonsingular linear complex. Geometers have studied linear complexes for a century or more without discovering any invariant subconfiguration, so one may hope that $GSp_{2n}(K)$ is a maximal subgroup of $GL_{2n}(K)$. However, the hope that this is so for all K is, as the proof of Theorem 3 makes clear, dashed as soon as one recalls that $Sp_{2n}(K) < SL_{2n}(K)$ [4, p. 11; 8, p. 224]. We prove (Theorem 3) that $GSp_{2n}(K)$ is maximal in $GL_{2n}(K)$ if and only if every element of K has an nth root in K. The containment $Sp_{2n}(K) < SL_{2n}(K)$ suggests that to obtain a "universal" maximality result associated with $\mathscr{F}$ one should take the stabilizer of $\mathscr{F}$ in $SL_{2n}(K)$. This is $SGSp_{2n}(K) = SL_{2n}(K) \cap GSp_{2n}(K)$. The notation is chosen to conform with the standard practice of writing SG to denote the intersection of a linear group G with the special linear group of the same dimension and over the same field; for parallel reasons $PSGSp_{2n}(K)$ will denote the quotient of $SGSp_{2n}(K)$ by its subgroup of scalar maps. We prove (Theorem 1) that $SGSp_{2n}(K)$ is maximal in $SL_{2n}(K)$, for all K. It then follows that $PSGSp_{2n}(K)$ is maximal in $PSL_{2n}(K)$: it is very satisfactory that the universal maximality result holds in the simple group $PSL_{2n}(K)$ [5, p. 39; 8, p. 182]. The condition $n > 1$ is always necessary since [8, p. 219] $SL_2(K) = Sp_2(K)$ and $GL_2(K) = GSp_2(K)$.

We also obtain conditions (Theorems 5, 6) for $Sp_{2n}(K)$ to be maximal in $SL_{2n}(K)$ and for $PSp_{2n}(K)$ to be maximal in $PSL_{2n}(K)$. The latter criterion is a little curious: it is that every nth root in K of 1 has a square root in K. All the maximality criteria take very simple forms for the finite, real and complex fields.

Since all nonsingular alternating bilinear forms on V are equivalent [4, p. 5] their corresponding $GSp_{2n}(K)$ form a conjugacy class of subgroups of $GL_{2n}(K)$. Since $SL_{2n}(K) \lhd GL_{2n}(K)$ the $SGSp_{2n}(K)$ form a set of isomorphic subgroups of $SL_{2n}(K)$ and a conjugacy class of subgroups of $GL_{2n}(K)$. This set of subgroups is the union of several conjugacy classes of subgroups of $SL_{2n}(K)$, each such conjugacy class corresponding to one orbit of symplectic polarities under $SL_{2n}(K)$. By the maximality Theorem 1, $SL_{2n}(K)$ acts primitively on each of these orbits. We classify them in Theorem 8, and show in particular that $SL_{2n}(K)$ is transitive on the set of symplectic polarities if and only if every element of K has an nth root in K, i.e., if and only if $GSp_{2n}(K)$ is maximal in $GL_{2n}(K)$.

2. The Maximality Theorems and Their Proofs

2.1. Take a base $e_1, e_2, \ldots, e_{2n}$ so that [4, p. 5] $B(x, y)$ has the standard canonical coordinate form

$$B(x, y) = \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i); \tag{2}$$

here $x = \sum_{l=1}^{2n} x_l e_l$.

We begin by giving, for completeness, a short direct proof of

Lemma 1. *Suppose that $A \in GL_{2n}(K)$ and has the property that $B(Ap, Aq) = 0$ for all pairs of vectors p, q in V such that $B(p, q) = 0$. Then $A \in GSp_{2n}(K)$.*

Proof. From (2) we have $B(e_l, e_m) = 0$ except when (l, m) is either an $(i, n+i)$ or an $(n+i, i)$ for some i less than or equal to n. Hence apart (possibly) from these exceptional (l, m) we have $B(Ae_l, Ae_m) = 0$. Now $B(Ax, Ay)$ is an alternating form. Hence

$$\begin{aligned} B(Ax, Ay) &= B\left(\sum_{l=1}^{2n} x_l A e_l, \sum_{m=1}^{2n} y_m A e_m\right) = \sum_{l,m=1}^{2n} x_l y_m B(Ae_l, Ae_m) \\ &= \sum_{i=1}^{n} \{x_i y_{n+i} B(Ae_i, Ae_{n+i}) + x_{n+i} y_i B(Ae_{n+i}, Ae_i)\} \\ &= \sum_{i=1}^{n} b_i (x_i y_{n+i} - x_{n+i} y_i), \end{aligned} \tag{3}$$

where $b_i = B(Ae_i, Ae_{n+i}) = -B(Ae_{n+i}, Ae_i)$. Let $u = \sum_{l=1}^{2n} e_l$. Then from (2) and (3) the zeros of the linear form $B(x, u) = \sum_{i=1}^{n} (x_i - x_{n+i})$ are zeros of $B(Ax, Au) = \sum_{i=1}^{n} b_i(x_i - x_{n+i})$. Since the former zeros are the vectors of a hyperplane of V and each hyperplane determines, up to scalar multiples, a unique linear form we see that there is an element λ in K such that $b_1 = b_2 = \cdots = b_n = \lambda$. Hence, by (2) and (3), $B(Ax, Ay) = \lambda B(x, y)$. Since $A \in GL_{2n}(K)$ the form $B(Ax, Ay)$ is nonsingular. Hence $\lambda \neq 0$ and thus, by (1), $A \in GSp_{2n}(K)$.

Note that since a totally isotropic 2-dimensional subspace can be taken as $\langle p, q \rangle$ where $B(p, q) = 0$ the Lemma shows that $GSp_{2n}(K)$ *is the group of* $\mathscr{F}$.

We can now prove

Theorem 1. *Suppose that $GSp_{2n}(K)$ is the group of symplectic similitudes of a nonsingular $2n$-dimensional alternating bilinear form over a field K and $SGSp_{2n}(K) = GSp_{2n}(K) \cap SL_{2n}(K)$. Then if $n > 1$ then $SGSp_{2n}(K)$ is a maximal subgroup of $SL_{2n}(K)$.*

Proof. If $0 \neq r \in V$ then a transvection centred on $\langle r \rangle$ is an element of $GL_{2n}(K)$ of the form [4, *p*. 4; 8, p. 178]

$$x \mapsto x + L(x)r, \tag{4}$$

where $L(x)$ is a non-zero linear form such that $L(r) = 0$. Now

$$L(x) = \sum_{i=1}^{n} (a_i x_i + a_{n+i} x_{n+i})$$

for some a_i, a_{n+i}, not all zero, in K. Hence, writing $s = \sum_{i=1}^{n}(a_i e_{n+i} - a_{n+i} e_i)$ we see, from (2), that $L(x) = B(x, s)$ and thus that any transvection centred on $\langle r \rangle$ has the form

$$x \mapsto x + B(x, s)r, \tag{5}$$

where $0 \neq s \in V$ and $B(r, s) = 0$. As is readily checked, this transvection is in $Sp_{2n}(K)$ if and only if [5, p. 10] $L(x)$ is a scalar multiple of $B(x, r)$, i.e., since $B(x, y)$ is nonsingular, if and only if $s \in \langle r \rangle$.

The map $x_1 \mapsto x_1 + x_n$, $x_l \mapsto x_l$ for $l > 1$ is in $SL_{2n}(K)$ if $n > 1$ and takes, by (2), $B(x, y)$ to $B(x, y) + (x_n y_{n+1} - x_{n+1} y_n)$. Hence it is not in $SGSp_{2n}(K)$ and so $SGSp_{2n}(K) < SL_{2n}(K)$. Suppose that

$$SGSp_{2n}(K) < L \leqslant SL_{2n}(K). \tag{6}$$

Take $C \in L \backslash SGSp_{2n}(K)$. Then $C \notin GSp_{2n}(K)$ and so, by Lemma 1, there are p, q in V such that $B(p, q) = 0$ and $B(Cp, Cq) \neq 0$. Thus $Cq \neq 0$ and hence $q \neq 0$. Hence the map

$$T : x \mapsto x + B(x, q)q$$

is a transvection of $Sp_{2n}(K)$. Since $Sp_{2n}(K) \leqslant SL_{2n}(K)$ [8, p. 224] we have $Sp_{2n}(K) \leqslant SGSp_{2n}(K)$. Hence, by (6), all sympletic transvections are in L. Thus $CTC^{-1} \in L$. Now

$$CTC^{-1}x = C(C^{-1}x + B(C^{-1}x, q)q) = x + B(C^{-1}x, q)Cq. \tag{7}$$

By (4), CTC^{-1} is a transvection centred on $\langle Cq \rangle$, and, by the remark after (5), it is in $Sp_{2n}(K)$ if and only if $B(C^{-1}x, q) = \mu B(x, Cq)$ for some element μ of K. If this is so, then, on taking Cp for x, we have

$$0 = B(p, q) = \mu B(Cp, Cq)$$

and thus, since $B(Cp, Cq) \neq 0$, it follows that $\mu = 0$ and thus, by (7), that CTC^{-1} is the identity: a contradiction. Hence $CTC^{-1} \notin Sp_{2n}(K)$ and L contains a non-symplectic transvection $S = CTC^{-1}$.

Suppose, now, that S is given by (5) with $r = Cq$. By the remarks after (5) we have $s \notin \langle r \rangle$ and $B(r, s) = 0$. Thus $\langle r, s \rangle$ is 2-dimensional and totally isotropic. If S^* is any non-symplectic transvection then, similarly, $S^*x = x + B(x, s^*)r^*$ where $\langle r^*, s^* \rangle$ is 2-dimensional and totally isotropic. The linear map from $\langle r, s \rangle$ to $\langle r^*, s^* \rangle$ determined by $r \mapsto r^*$, $s \mapsto s^*$ trivially takes the restriction of $B(x, y)$ to $\langle r, s \rangle$ to that to $\langle r^*, s^* \rangle$, and hence by Witt's theorem [2, p. 71], extends to any element A in $Sp_{2n}(K) \leqslant SGSp_{2n}(K)$. Thus [8, p. 218] $ASA^{-1} \in L$ by (6). Arguing as in (7) and using (5) we see that

$$\begin{aligned} ASA^{-1}x &= x + B(A^{-1}x, s)Ar = x + B(x, As)Ar \\ &= x + B(x, s^*)r^*. \end{aligned}$$

Hence $S^* = ASA^{-1} \in L$. Thus L contains all non-symplectic transvections. Since all symplectic transvections are in L all transvections are in L. The transvections generate $SL_{2n}(K)$ [8, p. 179]. Hence $L = SL_{2n}(K)$ and, by (6), we conclude that $SGSp_{2n}(K)$ is a maximal subgroup of $SL_{2n}(K)$.

As an almost immediate corollary we have

THEOREM 2. *If $n > 1$ then $PSGSp_{2n}(K)$, the quotient group of $SGSp_{2n}(K)$ by its subgroup of scalar maps, is a maximal subgroup of $PSL_{2n}(K)$.*

Proof. Let Z be the subgroup of scalar matrices of $SL_{2n}(K)$. From (2) every scalar map is in $GSp_{2n}(K)$ and so Z is the subgroup of scalars of $SGSp_{2n}(K)$. Hence $PSL_{2n}(K) = SL_{2n}(K)/Z$ and $PSGSp_{2n}(K) = SGSp_{2n}(K)/Z$. Moreover, by a standard homomorphism theorem, any subgroup of $PSL_{2n}(K)$ containing $PSGSp_{2n}(K)$ has the form M/Z, where M is a subgroup of $SL_{2n}(K)$ containing $SGSp_{2n}(K)$. By Theorem 1, M is either $SGSp_{2n}(K)$ or $SL_{2n}(K)$ and so M/Z is either $PSGSp_{2n}(K)$ or $PSL_{2n}(K)$. Hence $PSGSp_{2n}(K)$ is maximal in $PSL_{2n}(K)$.

2.2. Before proving our next result it is convenient to introduce some notation.

If $0 \neq \lambda \in K$ then let $D(\lambda)$ be the member of $GL_{2n}(K)$ given by

$$D(\lambda) = \text{diag.}(\lambda, \lambda, \ldots, \lambda, 1, 1, \ldots, 1), \tag{8}$$

the first n diagonal entries being λ. From (2)

$$B(D(\lambda)x, D(\lambda)y) = \lambda B(x, y), \tag{9}$$

and so $D(\lambda) \in GSp_{2n}(K)$. Thus the set of the $D(\lambda)$ is a subgroup $\mathscr{D}$ of $GSp_{2n}(K)$. By (9) the only member of $\mathscr{D} \cap Sp_{2n}(K)$ is the identity map $D(1)$. Further, by (1) and (9), if $A \in GSp_{2n}(K)$ then $[D(\lambda)]^{-1}A \in Sp_{2n}(K)$ for some non-zero λ in K. Clearly $Sp_{2n}(K) \trianglelefteq GSp_{2n}(K)$.

Hence

$$GSp_{2n}(K) = \mathscr{D} \cdot Sp_{2n}(K), \tag{10}$$

the product being semidirect.

We can now prove

THEOREM 3. *If $n > 1$ then $GSp_{2n}(K)$ is a maximal subgroup of $GL_{2n}(K)$ if and only if every member of K has an nth root in K.*

Proof. Since $SL_{2n}(K) \trianglelefteq GL_{2n}(K)$ and $Sp_{2n}(K) \leqslant SL_{2n}(K)$ we see, by (10), that $\mathscr{D} \cdot SL_{2n}(K)$ is a subgroup of $GL_{2n}(K)$ containing $GSp_{2n}(K)$. The first sentence of the second paragraph of the proof of Theorem 1 shows that there are elements of $SL_{2n}(K)$ that are not in $GSp_{2n}(K)$. Hence

$$GSp_{2n}(K) < \mathscr{D} \cdot SL_{2n}(K) \leqslant GL_{2n}(K). \tag{11}$$

Suppose that $GSp_{2n}(K) < L \leqslant GL_{2n}(K)$. Take $C \in L \backslash GSp_{2n}(K)$. A repetition of the argument in the proof of Theorem 1 following (6), noting that $L > SGSp_{2n}(K)$, shows that L contains all transvections. Hence $L \geqslant SL_{2n}(K)$. Now $L \geqslant \mathscr{D}$ by (10). Hence $L \geqslant \mathscr{D} \cdot SL_{2n}(K)$; i.e., every subgroup of $GL_{2n}(K)$ properly containing $GSp_{2n}(K)$ contains $\mathscr{D} \cdot SL_{2n}(K)$. Hence, by (11), $GSp_{2n}(K)$ is maximal in $GL_{2n}(K)$ if and only if $\mathscr{D} \cdot SL_{2n}(K) = GL_{2n}(K)$.

Suppose that this is so and $0 \neq \mu \in K$. Then diag.$(\mu, 1, 1, \ldots, 1) \in GL_{2n}(K)$ and so can be written as a product $D(\lambda)E$ for some element $\lambda \neq 0$ of K and some element E of $SL_{2n}(K)$. Taking determinants and using (8) we have $\mu = \lambda^n \cdot 1$, and so μ has nth root λ in K. Clearly 0 is an nth root of 0, and thus every element of K has an nth root in K. Conversely, suppose that each element of K has an nth root. Let $F \in GL_{2n}(K)$ and suppose that $\det F = \mu \neq 0$. Then there is a non-zero element λ of K such that $\mu = \lambda^n$ and hence, from (8), $\det[D(\lambda)]^{-1}F] = 1$. Thus $[D(\lambda)]^{-1}F \in SL_{2n}(K)$, i.e., $F \in \mathscr{D} \cdot SL_{2n}(K)$. Hence $GL_{2n}(K) = \mathscr{D} \cdot SL_{2n}(K)$. The theorem follows.

A repetition of the argument used in the proof of Theorem 2 with $SGSp_{2n}(K)$, $SL_{2n}(K)$, Z replaced respectively by $GSp_{2n}(K)$, $GL_{2n}(K)$, the subgroup of scalars of $GL_{2n}(K)$, shows that $PGSp_{2n}(K)$ is maximal in $PGL_{2n}(K)$ if and only if $GSp_{2n}(K)$ is maximal in $GL_{2n}(K)$. We have

THEOREM 4. *If $n > 1$ then $PGSp_{2n}(K)$, the quotient group of $GSp_{2n}(K)$ by its subgroup of scalar maps, is a maximal subgroup of $PGL_{2n}(K)$ if and only if every member of K has an nth root in K.*

In particular, *$GSp_{2n}(K)$ is maximal in $GL_{2n}(K)$ and $PGSp_{2n}(K)$ is maximal in $PGL_{2n}(K)$ when* (i) *$n > 1$ and K is algebraically closed, or* (ii) *K is a perfect field of characteristic $p > 0$ and n is a positive integral power of p*: in perfect fields of characteristic $p > 0$ one may repeatedly take pth roots.

2.3. It is possible for $Sp_{2n}(K)$ to be maximal in $SL_{2n}(K)$ and for $PSp_{2n}(K)$ to be maximal in $PSL_{2n}(K)$. Let

$$\Lambda = \mathscr{D} \cap SL_{2n}(K) = \{D(\lambda) : \lambda^n = 1\}, \tag{12}$$

from (8). Since $Sp_{2n}(K) \leqslant SL_{2n}(K)$ we have, by (10), (12) and Dedekind's rule,

$$SGSp_{2n}(K) = \Lambda \cdot Sp_{2n}(K), \tag{13}$$

the product being semidirect.

Hence, by Theorem 1, $Sp_{2n}(K)$ is maximal in $SL_{2n}(K)$ if and only if $Sp_{2n}(K) = SGSp_{2n}(K)$. By (13) this occurs if and only if $\Lambda \leqslant Sp_{2n}(K)$, which, because the product is semidirect, is so if and only if $| \Lambda | = 1$. We have, from (12),

THEOREM 5. *If $n > 1$ then $Sp_{2n}(K)$ is a maximal subgroup of $SL_{2n}(K)$ if and only if the only nth root of 1 in K is 1.*

In particular, *if K has characteristic $p > 0$ and n is a positive integral power of p then $Sp_{2n}(K)$ is maximal in $SL_{2n}(K)$*: in this case if $\lambda \in K$ then $\lambda^n - 1 = (\lambda - 1)^n$.

One cannot immediately deduce from Theorem 5 the condition that $PSp_{2n}(K)$ be maximal in $PSL_{2n}(K)$ since not every scalar map of $SL_{2n}(K)$ is in $Sp_{2n}(K)$. Let Y be the subgroup of scalar maps of $SL_{2n}(K)$. Then $PSL_{2n}(K) = SL_{2n}(K)/Y$ and $PSp_{2n}(K) = Sp_{2n}(K)/Y \cap Sp_{2n}(K)$. Hence $PSp_{2n}(K) \simeq Y \cdot Sp_{2n}(K)/Y$. By the standard homomorphism theorem there is a bijective correspondence between the set of subgroups of $PSL_{2n}(K)$ containing $Y \cdot Sp_{2n}(K)/Y$ and the set of subgroups of $SL_{2n}(K)$ containing $Y \cdot Sp_{2n}(K)$. Hence $PSp_{2n}(K)$ is maximal in $PSL_{2n}(K)$ if and only if $Y \cdot Sp_{2n}(K)$ is maximal in $SL_{2n}(K)$. From (2), $Y \leqslant GSp_{2n}(K)$ and hence $Y \cdot Sp_{2n}(K) \leqslant SGSp_{2n}(K)$. Hence, from Theorem 1, $Y \cdot Sp_{2n}(K)$ is maximal in $SL_{2n}(K)$ if and only if $Y \cdot Sp_{2n}(K) = SGSp_{2n}(K)$, which, by (13), occurs if and only if $\Lambda \leqslant Y \cdot Sp_{2n}(K) = Sp_{2n}(K) \cdot Y$.

Suppose that this is so and $\lambda \in K$ with $\lambda^n = 1$. Then $D(\lambda)$ can be written as a product AP where $A \in Sp_{2n}(K)$ and $P = \text{diag.}(\mu, \mu, \ldots, \mu)$ for some element $\mu \neq 0$ of K. Then, using (9) and (2),

$$\lambda B(x, y) = B(D(\lambda)x, D(\lambda)y) = B(APx, APy) = B(Px, Py) = \mu^2 B(x, y).$$

Hence $\lambda = \mu^2$, i.e., every nth root of 1 in K has a square root in K. Conversely, suppose that every nth root of 1 has a square root. Let $\lambda^n = 1$ so that $\lambda = \mu^2$ for some element μ in K. Then $D(\lambda) = D(\mu^2) = AP$ where $A = \text{diag.}(\mu, \mu, \ldots, \mu, \mu^{-1}, \mu^{-1}, \ldots, \mu^{-1})$, the first n entries being μ, and $P = \text{diag}(\mu, \mu, \ldots, \mu)$. From (2), $A \in Sp_{2n}(K)$. Further det $P = \mu^{2n} = \lambda^n = 1$ and so $P \in SL_{2n}(K)$ and thus $P \in Y$. Hence $D(\lambda) \in Sp_{2n}(Y) \cdot Y$ and so $\Lambda \leqslant Sp_{2n}(K) \cdot Y$. We obtain

THEOREM 6. *If $n > 1$ then $PSp_{2n}(K)$ is a maximal subgroup of $PSL_{2n}(K)$ if and only if every nth root of 1 in K has a sqauare root in K.*

In particular if K is algebraically closed or is a perfect field of characteristic 2 *then* $PSp_{2n}(K)$ *is maximal in* $PSL_{2n}(K)$.

2.4. Of particular interest to geometers and groups theorists are the cases when K is $\mathbb{C}$, the complex field, or $\mathbb{R}$, the real field, or $GF(q)$, a finite field of size q. If $K = \mathbb{C}$ then 1 has n distinct nth roots. Hence, by Theorem 5 and the remarks after Theorems 4, 6 we see that *if* $n > 1$ *then* $GSp_{2n}(\mathbb{C})$ *is maximal in* $GL_{2n}(\mathbb{C})$, $PGSp_{2n}(\mathbb{C})$ *is maximal in* $PGL_{2n}(\mathbb{C})$, $PSp_{2n}(\mathbb{C})$ *is maximal in* $PSL_{2n}(\mathbb{C})$, *but* $Sp_{2n}(\mathbb{C})$ *is not maximal in* $SL_{2n}(\mathbb{C})$. If $K = \mathbb{R}$ then nth roots exist for all elements if n is odd in which case 1 is the only nth root of 1. If n is even then -1 is an nth root of 1 but is not a square. Hence, by Theorems 3–6 we see that *if* $n > 1$ *and* n *is odd then* $GSp_{2n}(\mathbb{R})$ *is maximal in* $GL_{2n}(\mathbb{R})$, $PGSp_{2n}(\mathbb{R})$ *is maximal in* $PGL_{2n}(\mathbb{R})$, $Sp_{2n}(\mathbb{R})$ *is maximal in* $SL_{2n}(\mathbb{R})$, *and* $PSp_{2n}(\mathbb{R})$ *is maximal in* $PSL_{2n}(\mathbb{R})$; but when n is even none of these maximality statements holds.

Now suppose that $K = GF(q)$ and that $d = (q - 1, n)$, the highest common factor of $q - 1$ and n. The multiplicative group K^* of K is [3, p. 13] cyclic of order $q - 1$: let g be a generator. Every element of K has an nth root in K if and only if g has an nth root. This is so if and only if there is an integer x such that $(g^x)^n = g$, i.e., $g^{xn-1} = 1$. This is so if and only if there are integers x, y such that $xn - 1 = y(q - 1)$, i.e., $xn - y(q - 1) = 1$. This is so if and only if $d = 1$. Further, g^r is an nth root of 1 if and only if $g^{rn} = 1$. This is so if and only if $(q - 1) \mid rn$, which is so if and only if r is a multiple of $(q - 1)/d$. Thus 1 has d nth roots. Each of these is a square if and only if g^s is a square, where $s = (q - 1)/d$. This is so if and only if there is an integer u such that $g^s = (g^u)^2$, i.e., $g^{s-2u} = 1$ which occurs if and only if there are integers u, v such that $s - 2u = v(q - 1)$, i.e., $s = 2u + (q - 1)v$. Such u, v exist if and only if s is a multiple of $(2, q - 1)$. If q is even this is automatically so. If q is odd then $(2, q - 1) = 2$ so the condition is that $(q - 1)/d$ be even. From Theorems 3–6 we obtain

THEOREM 7. *Suppose that* $K = GF(q)$ *and* $n > 1$. *Then*:

(i) $GSp_{2n}(q)$ *is maximal in* $GL_{2n}(q)$ *if and only if* $(q - 1, n) = 1$;

(ii) $PGSp_{2n}(q)$ *is maximal in* $PGL_{2n}(q)$ *if and only if* $(q - 1, n) = 1$;

(iii) $Sp_{2n}(q)$ *is maximal in* $SL_{2n}(q)$ *if and only if* $(q - 1, n) = 1$;

(iv) $PSp_{2n}(q)$ *is maximal in* $PSL_{2n}(q)$ *if and only if either* q *is even or* q *is odd and* $(q - 1)/(q - 1, n)$ *is even.*

In particular $PSp_{2n}(q)$ *is maximal in* $PSL_{2n}(q)$ *when* n *and* q *are both odd.*

Mwene [12, pp. 79–80] has obtained a complete list of the maximal subgroups of $PSL_4(q)$ when q is even, and $Sp_4(q)$ occurs in his list. When q is even $Sp_4(q) \cong PSp_4(q)$: if in the notation of Section 2.3, diag. $(\mu, \mu, \ldots, \mu) \in Y \cap Sp_4(q)$ then, by (2), $\mu^2 = 1$ and so $\mu = 1$. Thus Theorem 7(iv) with $n = 2$ and q even is in agreement with [12]. Theorem 7(iii) shows that $Sp_{2n}(2)$ is maximal in $SL_{2n}(2)$ for $n > 1$. This is proved by an alternative method in [6].

3. The Action of $SL_{2n}(K)$ on Symplectic Polarities

We prove

Theorem 8. *Suppose that K^* is the multiplicative group of a field K and $(K^*)^n$ is the set of all the nth powers of the members of K^*. Then the set of the orbits under $SL_{2n}(K)$ of the nonsingular symplectic polarities on a $2n$-dimensional vector space over K is in bijective correspondence with $K^*/(K^*)^n$.*

Proof. The result is trivial if $n = 1$ since there is only one symplectic polarity, so assume that $n > 1$. Choose and fix a base of V; let x have coordinates $(x_1, x_2, \ldots, x_{2n})$. If $0 \neq \mu \in K$ write

$$E(\mu) = \text{diag.}(\mu, 1, 1, \ldots, 1), \tag{14}$$

$$B_\mu(x, y) = \mu(x_1 y_{n+1} - x_{n+1} y_1) + \sum_{i=2}^{n} (x_i y_{n+i} - x_{n+i} y_i). \tag{15}$$

Thus

$$B_\mu(x, y) = B_1(E(\mu)x, E(\mu)y). \tag{16}$$

Let $B(x, y)$ be a nonsingular alternating form. By the equivalence of such forms [4, p. 5] there is an element P of $GL_{2n}(K)$ such that

$$B(Px, Py) = B_1(x, y). \tag{17}$$

This is equivalent to taking the canonical coordinate form in (2). Note that (9) now holds with B_1 in place of B. Suppose that $\det P = \nu^{-1}$ and let $Q = PE(\nu)$. From (14), $\det Q = \nu^{-1} \cdot \nu = 1$ so that $Q \in SL_{2n}(K)$. From (17) and (16),

$$B(Qx, Qy) = B(PE(\nu)x, \quad PE(\nu)y) = B_1(E(\nu)x, \quad E(\nu)y) = B_\nu(x, y).$$

Thus each orbit of nonsingular alternating bilinear forms under $SL_{2n}(K)$ contains some $B_\mu(x, y)$ with $\mu \neq 0$: each such $B_\mu(x, y)$ is a nonsingular alternating bilinear form by (15). A nonsingular alternating form determines a unique symplectic polarity, and the polarity determines the form uniquely to within a scalar factor; see Lemma 1. Clearly, from (15), since $n > 1$ if $\mu, \nu \in K^*$ and $\mu \neq \nu$ then the polarities associated with $B_\mu(x, y)$ and $B_\nu(x, y)$ are different. Hence the theorem will be proved once it has been shown that the polarities associated with $B_\mu(x, y)$ and $B_\nu(x, y)$ are in the same orbit under $SL_{2n}(K)$ if and only if $\mu\nu^{-1} \in (K^*)^n$, i.e., μ and ν are in the same coset of $(K^*)^n$ in K^*.

Suppose, first, that the polarities associated with $B_\mu(x, y)$ and $B_\nu(x, y)$ are in the same orbit. Then there is an element R of $SL_{2n}(K)$ such that

$$B_\mu(Rx, Ry) = \lambda B_\nu(x, y) \tag{18}$$

for some element λ of K^*. Hence, by (16) and (9) with B_1 in place of B,

$$B_1(E(\mu)Rx, E(\mu)Ry) = \lambda B_1(E(\nu)x, E(\nu)y) = B_1(D(\lambda)\, E(\nu)x, D(\lambda)\, E(\nu)y).$$

Thus $E(\mu)\, R[D(\lambda)\, E(\nu)]^{-1} \in Sp_{2n}(K) \leqslant SL_{2n}(K)$, where $Sp_{2n}(K)$ is the symplectic group of $B_1(x, y)$. Taking determinants and using (8) and (14) yields $\mu \cdot 1 \cdot \lambda^{-n} \cdot \nu^{-1} = 1$. Hence $\mu\nu^{-1} = \lambda^n$ and thus $\mu\nu^{-1} \in (K^*)^n$.

Conversely, suppose that $\mu\nu^{-1} \in (K^*)^n$. Then $\mu\nu^{-1} = \lambda^n$ for some element λ in K^*. Let $S = [E(\mu)]^{-1}\, D(\lambda)\, E(\nu)$. Then $\det S = \mu^{-1} \cdot \lambda^n \cdot \nu = 1$ so that $S \in SL_{2n}(K)$. Further, by successive use of (16), (9) with B_1 in place of B, and (16),

$$\begin{aligned}\lambda B_\nu(x, y) &= \lambda B_1(E(\nu)x, E(\nu)y) = \lambda B_1([D(\lambda)]^{-1}E(\mu)Sx, [D(\lambda)]^{-1}E(\mu)Sy) \\ &= B_1(E(\mu)Sx, E(\mu)Sy) = B_\mu(Sx, Sy).\end{aligned}$$

Hence the polarities associated with $B_\mu(x, y)$ and $B_\nu(x, y)$ are in the same orbit under $SL_{2n}(K)$. The theorem follows.

Note that if $B_\mu(x, y)$ and $B_\nu(x, y)$ are in the same orbit of alternating forms under $SL_{2n}(K)$ then (18) must hold with $\lambda = 1$. The argument immediately following (18) then yields $\mu\nu^{-1} = 1^n$. Thus $\mu = \nu$. Hence from the proof of Theorem 8 we have

COROLLARY 1. (i) *The set of the orbits under $SL_{2n}(K)$ of the nonsingular alternating bilinear forms on a $2n$-dimensional vector space over K is in bijective correspondence with K^*.*

(ii) *Under $SL_{2n}(K)$ each nonsingular alternating bilinear form is equivalent to one and only one $B_\mu(x, y)$.*

From Theorem 8 we immediately deduce

COROLLARY 2. *$SL_{2n}(K)$ is transitive on the nonsingular symplectic polarities on a $2n$-dimensional vector space over K if and only if every element of K has an nth root in K.*

Referring to the remarks after Theorem 4 and in Section 2.4 we see that *$SL_{2n}(K)$ acts transitively on the nonsingular symplectic polarities when*

(i) *K is algebraically closed,*

(ii) *$K = \mathbb{R}$ and n is odd,*

(iii) *$K = GF(q)$ and $(q-1, n) = 1$.*

Theorem 8 has an immediate analogue for the orbits of linear complexes in projective space under $PSL_{2n}(K)$, and Corollary 1 has one for the equivalence under $SL_{2n}(K)$ of nonsingular alternating matrices: the reader will perceive these at a glance. I have been unable to find a reference to Corollary 1(ii) in the literature.

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